Destabilizing Effects of Market Size in the Dynamics of Innovation

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Introduction

Market Size Effect on the Dynamics of Innovation

- Many existing studies on the market size effect on innovation & long-run growth.

- Little is known about the market size effect on **patterns of fluctuations** in innovation.

- In existing models of **endogenous innovation cycles**, market size merely alters the amplitude of fluctuations without affecting the nature of fluctuations.
  - Due to **CES homothetic demand system** for innovated products, monopolistically competitive firms sell their products at an **exogenous markup rate**
  - **Procompetitive effect** of market size is missing

- We introduce ***the procompetitive effect*** into a model of endogenous innovation cycles using a ***more general homothetic demand system***
  - Judd (1985; section 4)
  - H.S.A. demand system

- We show: Through **the procompetitive effect,**

  A larger market size/innovation cost ratio

  → the innovators face **more elastic demand** & sell their products at **lower markup rates**

  → *Destabilizing effects* in the dynamics of innovation

Judd (1985) Models of Endogenous Innovation Cycles

Dynamic Dixit-Stiglitz model, with symmetric CES technology, where the masses of both competitively and monopolistically supplied input varieties change over time through *innovation, diffusion, and obsolescence*

- Innovators pay a one-time innovation cost to introduce a new variety.
- Innovators keep monopoly power for a limited time. Then, their innovations become supplied competitively. That's when innovations in the recent past reach their full potential, causing the market to saturate with a delay.
- This creates the force for temporary clustering of innovations.
- All existing varieties, subject to idiosyncratic obsolescence shocks.

Judd (1985; Sec.3); *Continuous time* and monopoly lasting for $0 < T < \infty$

- *Delayed differential equation* (with an infinite dimensional state space)
- For $T > T_c > 0$, the economy alternates between the phases of active innovation and of no innovation along any equilibrium path for almost all initial conditions.

Judd (1985; Sec.4); a simple yet rich set of dynamics, incl. ***endogenous fluctuations***

- *Discrete time* and *one period monopoly* for analytical tractability
- **1D state space** (the mass of competitive varieties inherited from the past determines how saturated the market is)
- Dynamics governed by a **1D PWL (skewed V-) map**, fully characterized
- Properties depend on the "**delayed impact of innovations**", a constant number, determined solely by (exogenously) constant price elasticity, hence invariant of the market size and innovation cost.

Symmetric H.S.A. (Homothetic with a Single Aggregator) Demand System with Gross Substitutes

Market Share of each input depends *solely* on its single relative price (=its own price/the *common* price aggregator)

$$\frac{p_t(\omega)}{P(\mathbf{p}_t)}\frac{\partial P(\mathbf{p}_t)}{\partial p_t(\omega)} = s\left(\frac{p_t(\omega)}{A(\mathbf{p}_t)}\right) \qquad \text{where} \qquad \int_{\Omega_t} s\left(\frac{p_t(\omega)}{A(\mathbf{p}_t)}\right) d\omega \equiv 1$$

- $s: \mathbb{R}_{++} \to \mathbb{R}_+$: **the market share function**, decreasing in the **relative price,** $s(\infty) = 0$.
  If $\bar{z} \equiv \inf\{z > 0 | s(z) = 0\} < \infty$, $\bar{z}A(\mathbf{p})$ is the **choke price**.
- $A(\mathbf{p})$: the **common price aggregator** defined implicitly by **the adding-up constraint,** $\int_{\Omega_t} s\left(\frac{p_t(\omega)}{A(\mathbf{p}_t)}\right) d\omega \equiv 1$

By construction, market shares add up to one; $A(\mathbf{p})$ linear homogenous in $\mathbf{p}$ for a fixed Ω. A larger Ω reduces $A(\mathbf{p})$.

CES if $s(z) \propto z^{1-\sigma}$ with $\sigma > 1$; translog if $s(z) \propto -\gamma \ln(z/\bar{z})$; CoPaTh if $s(z) \propto \gamma\left[1 - (z/\bar{z})^{\frac{1-\rho}{\rho}}\right]^{\frac{\rho}{1-\rho}}$ with $\rho \in (0,1)$.

Unit Cost Function: $P(\mathbf{p}) \propto A(\mathbf{p}) \exp\left\{-\int_{\Omega} \left[\int_{p_\omega/A(\mathbf{p})}^{\bar{z}} \frac{s(\xi)}{\xi} d\xi\right] d\omega\right\}$

Matsuyama-Ushchev (2017) proved that $P(\mathbf{p})$ is quasi-concave, and $A(\mathbf{p}) \neq cP(\mathbf{p})$ for any $c > 0$, unless CES

✓ $A(\mathbf{p})$, the inverse measure of *competitive pressures*, captures the *cross-effects* in the demand system

✓ $P(\mathbf{p})$, the inverse measure of TFP, capturing the *productivity consequences* of price changes

Monopolistic Competition under H.S.A.

Price elasticity of demand for an input depends only on its relative price, $\zeta\left(\frac{p_t(\omega)}{A(\mathbf{p}_t)}\right)$, where

$$\zeta(z) \equiv 1 - \frac{s'(z)z}{s(z)} > 1$$

- $\zeta(\cdot)$ is constant under CES, since $s(z) = \gamma(z)^{1-\sigma} \Leftrightarrow \zeta(z) = \sigma$.
- Under *increasing* $\zeta(\cdot)$, **Marshall's 2nd law of demand**, implying the **Procompetitive effect** of market size and entry

Main Results: Dynamics of Innovation in the Judd model under H.S.A. with the Procompetitive Effect

- *still* governed by a 1D-PWL (skewed V-) map, hence remain equally tractable.
- **Delayed impact of innovations**, *still* a constant number (with the same range of the value if $\zeta(\cdot)$ is increasing), but now depends on the market size/innovation cost ratio.
- A larger market size/innovation cost ratio, by reducing the markup rate via the procompetitive effect, increases the delayed impact, with **destabilizing** effects in the dynamics of innovation under **the two sets of sufficient conditions**
  - Log-concavity of $\zeta(\cdot) - 1$, i.e., if $\zeta(\cdot)$ is "not too convex."
  - Two parametric families with the choke prices, "Generalized Translog" and "Constant Pass-Through." Each contains CES as a limit case, and yet, the properties of the dynamical system exhibit *discontinuity* in the limit.
- In a multi-market extension, because innovation/entry activities fluctuate more in larger markets,
  - They are not always higher in larger markets.
  - The sale of each product is more volatile in larger markets.

Innovation Cycle under CES: Revisiting Judd (1985)

Time: $t \in \{0, 1,2, \ldots\}$

Representative Household: supply L units of labor (numeraire), consume the single perishable consumption good, C_t, with $P_t C_t = L$.
No need to specify intertemporal preferences. With no means to save in the model, the interest rate adjusts endogenously to force the household to spend its income each period.

Competitive Final Goods Producers: assemble differentiated inputs using symmetric CES with gross substitutes

$$C_t = Y_t = F(\mathbf{x}_t) = Z\left[\int_{\Omega_t} [x_t(\omega)]^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \quad (\sigma > 1),$$

Unit cost function:

$$P_t = P(\mathbf{p}_t) = \frac{1}{Z}\left[\int_{\Omega_t} [p_t(\omega)]^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}}$$

Demand for a Differentiated Input:

$$x_t(\omega) = \frac{[p_t(\omega)]^{-\sigma} L}{\int_{\Omega_t} [p_t(\omega')]^{1-\sigma} d\omega'}$$

Sets of differentiated inputs: change due to *innovation, diffusion, obsolescence*
$\Omega_t = \Omega_t^c + \Omega_t^m$: Set of all differentiated inputs available in t, partitioned into:

- Ω_t^m: Set of new inputs introduced & sold exclusively by the innovators *for one period*.
- Ω_t^c: Set of competitively supplied inputs, which were innovated before t.

Production & pricing of differentiated inputs: ψ units of labor per unit of each variety

$$p_t(\omega) = \psi = p^c, \qquad x_t^c(\omega) \equiv x_t^c \quad for\ \omega \in \Omega_t^c \subset \Omega_t$$

$$p_t(\omega) = \frac{\psi}{1 - 1/\sigma} \equiv p^m, \ \ x_t^m(\omega) \equiv x_t^m \quad for\ \omega \in \Omega_t^m \subset \Omega_t$$

$$\frac{p^c}{p^m} = 1 - \frac{1}{\sigma} < 1, \qquad \frac{x_t^c}{x_t^m} = \left(1 - \frac{1}{\sigma}\right)^{-\sigma} > 1, \qquad \frac{p^c x_t^c}{p^m x_t^m} = \left(1 - \frac{1}{\sigma}\right)^{1-\sigma} \equiv \theta \in (1, e)$$

Note: *in equilibrium,* each variety faces the demand curve,

$$x_t(\omega) = \frac{L\big(p_t(\omega)\big)^{-\sigma}}{V_t^c(p^c)^{1-\sigma} + V_t^m(p^m)^{1-\sigma}} = \frac{L}{V_t(\psi)^{1-\sigma}}\big(p_t(\omega)\big)^{-\sigma}$$

and

$$\frac{Y_t}{L} = \frac{1}{P_t} = Z[V_t^c(p_t^c)^{1-\sigma} + V_t^m(p_t^m)^{1-\sigma}]^{\frac{1}{\sigma-1}} = \frac{Z}{\psi}(V_t)^{\frac{1}{\sigma-1}}$$

where V_t^c (V_t^m) is the measure of Ω_t^c (Ω_t^m) and

$$V_t \equiv V_t^c + V_t^m/\theta$$

- One competitive variety equivalent to $\theta > 1$ monopolistic varieties. V_t: "competitive-equivalent" mass of varieties
- Impact of an innovation magnified by $\theta > 1$ *when* its innovator loses monopoly, reaching its full potential
- Past innovations more discouraging than contemporaneous ones to innovators → incentive for temporal clustering of innovations

Introduction of New Varieties: Innovation cost per variety, F

$$V_t^m \geq 0;\ (p^m - \psi)x_t^m = \frac{p^m x_t^m}{\sigma} = \frac{p^c x_t^c}{\sigma\theta} \leq F$$

Complementary Slackness: Either net profit or innovation is zero in equilibrium

Resource Constraint:

$$L = V_t^c(\psi x_t^c) + V_t^m(\psi x_t^m + F) = V_t^c(p^c x_t^c) + V_t^m(p^m x_t^m) = V_t(p^c x_t^c)$$

$$\Longrightarrow V_t \equiv V_t^c + \frac{V_t^m}{\theta} = max\left\{\frac{L}{\sigma\theta F}, V_t^c\right\} \Leftrightarrow V_t^m = max\left\{\frac{L}{\sigma F} - \theta V_t^c, 0\right\}$$

- When innovations are active ($V_t^m > 0$),
  - one competitive variety crowds out $\theta > 1$ innovations.
  - $x_t^c = \sigma\theta(F/\psi)$ and $x_t^m = (\sigma - 1)(F/\psi)$; *independent of* L, which affects only how much innovation takes place.
- $L/(\sigma\theta F)$: the saturation level of competitive varieties, which kills incentive to innovate.

Idiosyncratic Obsolescence Shock: $\delta \in (0,1)$, the survival rate

$$V_{t+1}^c = \delta(V_t^c + V_t^m) = \delta max\left\{\frac{L}{\sigma F} + (1-\theta)V_t^c, V_t^c\right\}$$

Define $n_t \equiv \left(\frac{\sigma\theta F}{L}\right) V_t^c$: the *market saturation rate* = the normalized mass of competitive varieties = the market share of the competitive varieties, when $n_t \leq 1$

Dynamical System

$$n_{t+1} = f(n_t) \equiv \begin{cases} f_L(n_t) \equiv & \delta(\theta + (1-\theta)n_t) \quad for\ n_t < 1 \\ f_H(n_t) \equiv & \delta n_t \quad for\ n_t > 1 \end{cases}$$

$\delta \in (0,1)$, Survival rate of each variety

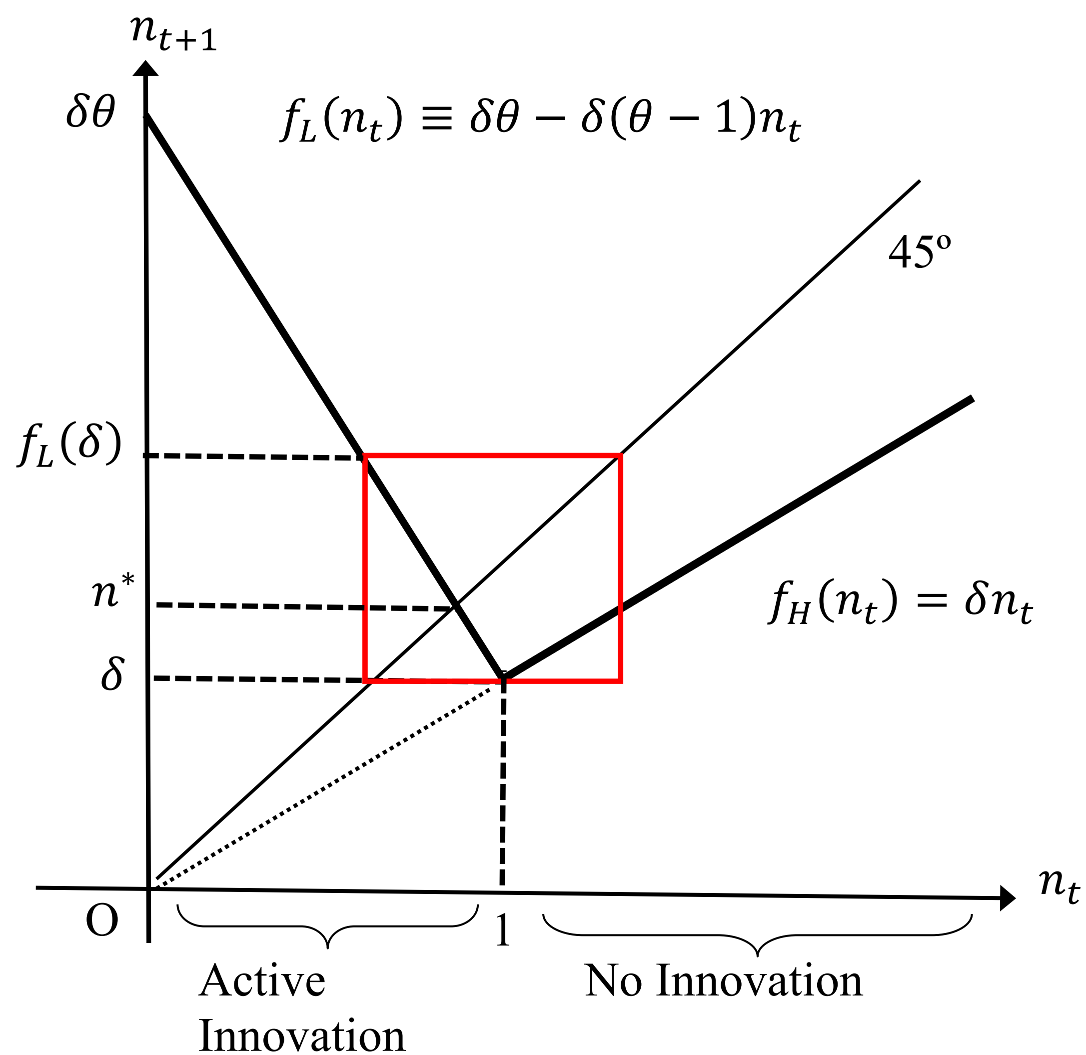
$\theta \equiv \left(1 - \frac{1}{\sigma}\right)^{1-\sigma} \in (1, e)$, increasing in σ; Market share multiplier due to the loss of monopoly power by its innovator

$\theta - 1 > 0$: *Delayed impact of innovations*

$$V_t^c = \frac{L}{\sigma\theta F} n_t;\ V_t^m = \frac{L}{\sigma F} max\{1 - n_t, 0\};$$

$$V_t \equiv V_t^c + \frac{V_t^m}{\theta} = \frac{L}{\sigma\theta F} max\{1, n_t\};$$

$$P_t = \frac{\psi}{Z}(V_t)^{\frac{1}{1-\sigma}};\ \frac{Y_t}{L} = \frac{1}{P_t} = (V_t)^{\frac{1}{\sigma-1}} \frac{Z}{\psi}$$



Key Features of Dynamical System

- 1D-PWL noninvertible (a skewed V-shaped) map
- θ depends solely on σ; *independent of* L, F *and* ψ.
- L/F merely affects the amplitude of fluctuations.

A *Unique* Attractor:

- *Stable steady state* for $\delta(\theta - 1) < 1$, globally attracting

For $\delta(\theta - 1) < 1$, unstable steady state, but the trajectory trapped into the red box.

- *Stable 2-cycle* for $\delta^2(\theta - 1) < 1 < \delta(\theta - 1)$, to which almost always converging
- *Robust chaotic attractor* with 2^m intervals (m = 0, 1,…) for $\delta^2(\theta - 1) > 1$

Effects of $\boldsymbol{\delta}$ (for $\theta = 2.5$)
(In courtesy of L. Gardini and I. Sushko)

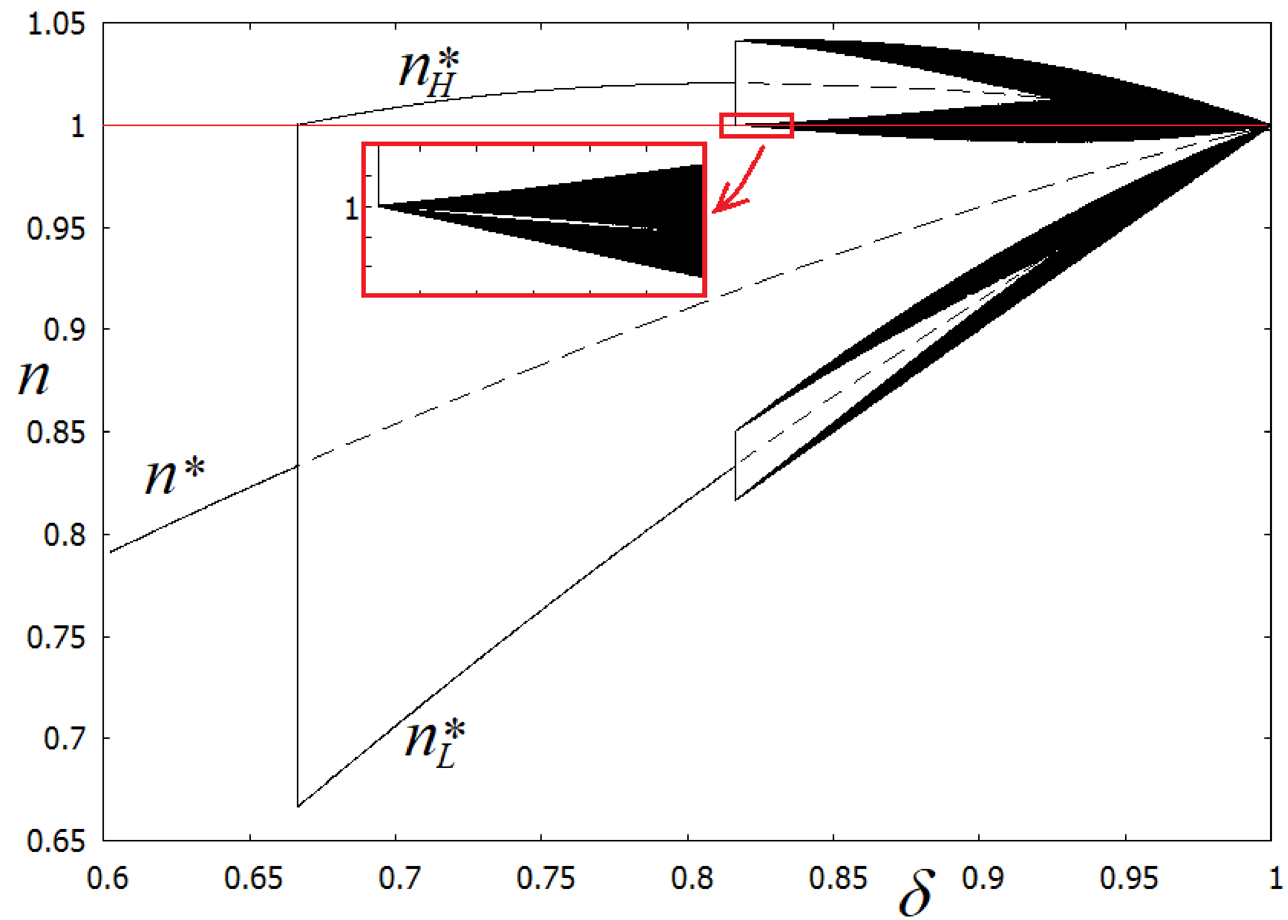
Notes:

✓ Most existing examples of chaos in econ are *not* attractors:
"Period three does *not* imply chaotic *attractors*."

✓ Most existing examples of chaotic attractors in econ are *not* robust,

since most applications use smooth dynamical systems.

Judd model has a robust chaotic attractor due to its regime-switching (nonsmooth) feature.

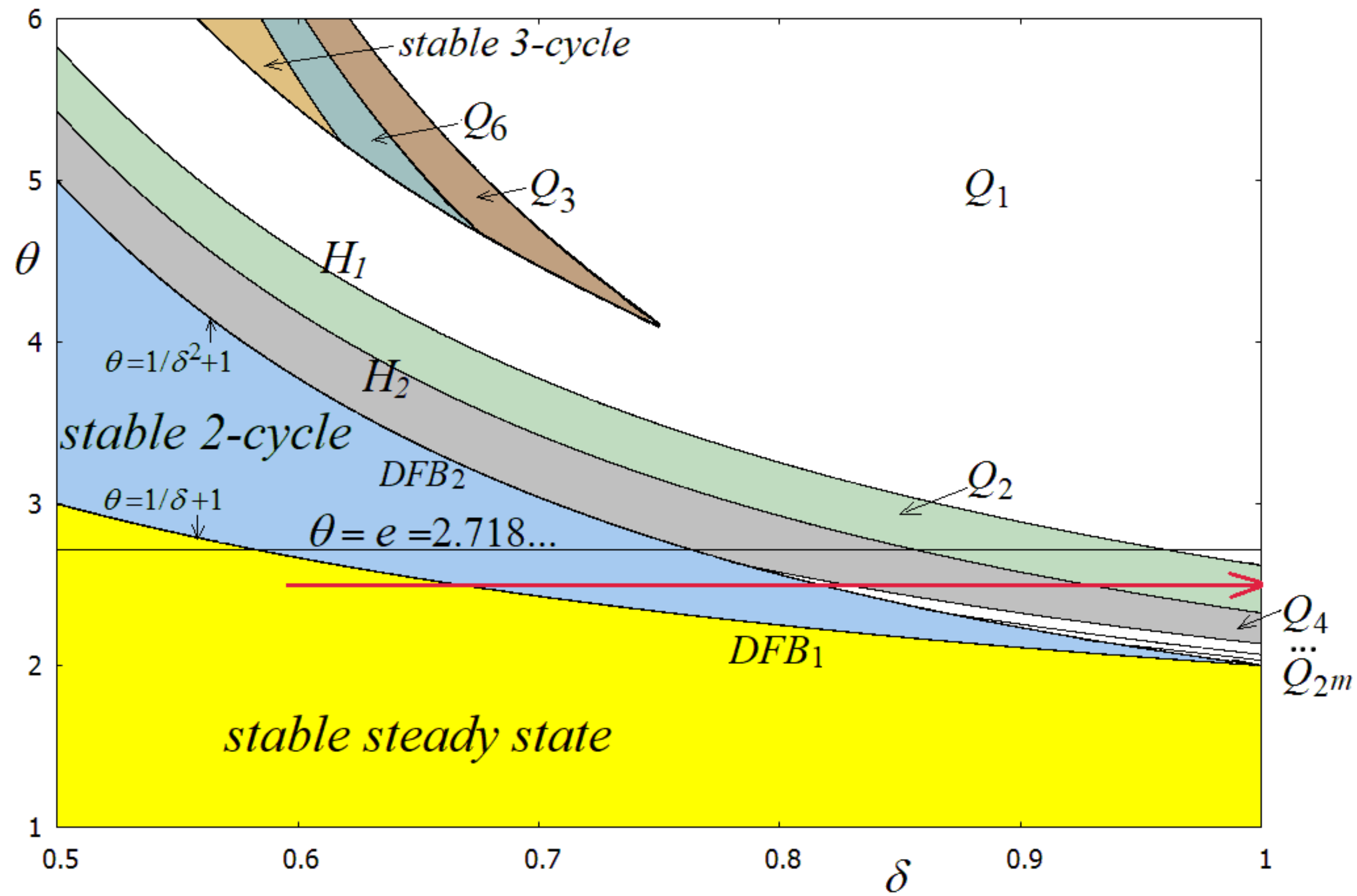


In the $(\boldsymbol{\delta}, \boldsymbol{\theta})$-plane Endogenous fluctuations with

- a higher $\sigma \Longrightarrow$ a higher θ (past innovations are more discouraging than contemporaneous innovations, stronger incentive for temporal clustering).
- a higher δ (more current innovation survives to discourage future innovation).

(In courtesy of L. Gardini and I. Sushko)

Note: In general, an attractor of a skewed V-shaped map could be a stable p-cycle or a robust chaotic attractor with p-intervals (where p can be any natural number.) But, this can be ruled out in the Judd model because of $\theta < e$.



Even with a stable steady state, slower convergence with a higher σ (θ) and/or a higher δ.

Innovation Cycle under H.S.A.

Symmetric H.S.A. (Homothetic with a Single Aggregator) Demand System with Gross Substitutes

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Pricing of Competitive Inputs: $p_t(\omega) = \psi = p^c$ for all $\omega \in \Omega_t^c$

Pricing of Monopolistic Inputs: $Max\ (p_t(\omega) - \psi)x_t(\omega) = \left(1 - \frac{\psi}{p_t(\omega)}\right) s\left(\frac{p_t(\omega)}{A(\mathbf{p}_t)}\right) L$

FOC: $\frac{p_t(\omega)}{A(\mathbf{p}_t)}\left[1 - \frac{1}{\zeta(p_t(\omega)/A(\mathbf{p}_t))}\right] = \frac{\psi}{A(\mathbf{p}_t)}$, where $\zeta(z)$ is the price elasticity, given by

$$\zeta(z) \equiv 1 - \frac{zs'(z)}{s(z)} > 1 \Leftrightarrow s(z) = \int_z^\infty \frac{\zeta(\tau) - 1}{\tau} d\tau > 0$$

(A1): $z\left(1 - \frac{1}{\zeta(z)}\right)$ is increasing for all $z \Leftrightarrow \frac{d\ln M(z)}{d\ln(z)} \equiv \frac{d\ln\zeta(z)/[\zeta(z)-1]}{d\ln(z)} < 1$ for all $z \Leftrightarrow \frac{s(z)}{\zeta(z)}$ is decreasing for all z

Under (A1), the LHS of FOC strictly increasing in $p_t(\omega)/A(\mathbf{p}_t) \rightarrow$ symmetric pricing: $p_t(\omega) = p_t^m$ for all $\omega \in \Omega_t^m$

Define $z_t^m \equiv p_t^m/A(\mathbf{p}_t)$; $z_t^c \equiv p_t^c/A(\mathbf{p}_t)$. Then,

Profit Maximization: $z_t^m\left[1 - \frac{1}{\zeta(z_t^m)}\right] = z_t^c \Leftrightarrow z_t^m = M(z_t^m)\, z_t^c$ z_t^m & z_t^c move together under (A1)

Maximized Profit: $\left(1 - \frac{z_t^c}{z_t^m}\right) s(z_t^m)L = \frac{s(z_t^m)}{\zeta(z_t^m)} L$ decreasing in z_t^m under (A1)

Adding Up Constraint: $V_t^m s(z_t^m) + V_t^c s(z_t^c) = 1.$

Innovation (Free Entry): Complementarity Slackness Condition.

$$V_t^m \geq 0;\ \frac{s(z_t^m)}{\zeta(z_t^m)} L \leq F$$

For $V_t^m > 0$,

$$z_t^m = \underline{z}^m;\ z_t^c = \underline{z}^c \equiv \underline{z}^m \left[1 - \frac{1}{\zeta(\underline{z}^m)}\right]$$

where

$$\frac{\zeta(\underline{z}^m)}{s(\underline{z}^m)} \equiv \frac{L}{F}$$

For $\zeta(0)F < s(0)L$,
$\underline{z}^m > 0$ well-defined; increasing in L/F under (A1).

From $V_t^m s(\underline{z}^m) + V_t^c s(\underline{z}^c) = 1$,

$$V_t^m = \theta\left(\frac{1}{s(\underline{z}^c)} - V_t^c\right) > 0, \text{where } \theta \equiv \frac{s(\underline{z}^c)}{s(\underline{z}^m)} > 1.$$

For $V_t^m = 0$, $z_t^m \geq \underline{z}^m$ & $z_t^c \geq \underline{z}^c$. Hence, $V_t^c = \frac{1}{s(z_t^c)} \geq \frac{1}{s(\underline{z}^c)}$ and,

$$V_t^m = max\left\{\theta\left(\frac{1}{s(\underline{z}^c)} - V_t^c\right), 0\right\} = max\left\{\frac{L}{\zeta(\underline{z}^m)F} - \theta V_t^c, 0\right\}$$

Idiosyncratic Obsolescence Shock: $\delta \in (0,1)$, the survival rate

$$V_{t+1}^{c} = \delta(V_t^c + V_t^m) = \delta max\left\{\frac{\theta}{s(\underline{z}^c)} + (1-\theta)V_t^c, V_t^c\right\}$$

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where

$$\theta \equiv \frac{s(\underline{z}^c)}{s(\underline{z}^m)} = \frac{s\left(\underline{z}^m\left[1 - \frac{1}{\zeta(\underline{z}^m)}\right]\right)}{s(\underline{z}^m)} > 1; \quad \frac{\zeta(\underline{z}^m)}{s(\underline{z}^m)} \equiv \frac{L}{F}$$

Notes:

- Dynamics are still governed by the same 1D PWL (skewed V-shape) map, as before.
- Its slopes still depend only on δ and θ, but θ now depends on $s(\cdot)$ and L/F.
- Under (A1), $\underline{z}^m$ and $\underline{z}^c$ are both increasing in L/F.

Procompetitive Effect with Marshall's 2nd Law of Demand

(A2): $\zeta'(\cdot) \geq 0$.

- (A2) $\Longrightarrow$ (A1).
- As its price goes up, the price elasticity of an input never declines. When the inequality in (A2) is weak (strict), we say that the weak (strong) form of Marshall's 2nd Law of Demand is satisfied.
- By imposing (A2), we depart from CES only to allow for the Procompetitive Effect, Strategic Complementarity in pricing, and (firm-level) Incomplete Pass-through, since

A strictly increasing (decreasing) $\zeta(z)$ implies

i) A larger market size has a pro-competitive (anti-competitive) effect, since $\zeta\left(\underline{z}^m\right)$ is strictly increasing (decreasing) in L/F.

F.O.C. under heterogeneous costs implies

ii) Pricing of monopolistic varieties are strategic complements (strategic substitutes).

$$\frac{d\ln p_t^m(\omega)}{d\ln A(\mathbf{p}_t)} = \frac{\Delta}{1+\Delta} > (<) 0$$

iii) Firm-level pass-through rate is less (more) than 100%

$$\frac{d\ln p_t^m(\omega)}{d\ln \psi(\omega)} = \frac{1}{1+\Delta} < (>) 1,$$

where $\Delta \equiv \frac{z_t(\omega)\zeta'(z_t(\omega))}{[\zeta(z_t(\omega))-1]\zeta(z_t(\omega))} = -\frac{d\ln M(z_t(\omega))}{d\ln(z_t(\omega))} > (<) 0$, and (A1) implies $1+\Delta > 0$.

The Two Main Propositions

Question #1: What is the range of θ? For CES, $\theta \in (1, e)$, where $e = 2.718 \ldots$

Proposition 1. Under the weak 2nd Law, $\theta \in (1, e)$, where $e = 2.718 \ldots$

Thus, the types of asymptotic behaviors observable are the same with CES.

Sketch of Proof: Under (A2), $\zeta(\cdot)$ is non-decreasing. Hence,

$$\theta \equiv \frac{s(\underline{z}^c)}{s(\underline{z}^m)} = \exp\left[\int_{\underline{z}^c}^{\underline{z}^m} \frac{\zeta(\tau) - 1}{\tau} d\tau\right] \leq \exp\left[\left(\zeta(\underline{z}^m) - 1\right) \int_{\underline{z}^c}^{\underline{z}^m} \frac{d\tau}{\tau}\right] =$$

$$\exp\left[\left(\zeta(\underline{z}^m) - 1\right) \log\left(\frac{\underline{z}^m}{\underline{z}^c}\right)\right] = \exp\left[\log\left(\frac{\underline{z}^c}{\underline{z}^m}\right)^{1-\zeta(\underline{z}^m)}\right] = \left(1 - \frac{1}{\zeta(\underline{z}^m)}\right)^{1-\zeta(\underline{z}^m)} < e$$

Intuition: Under (A2), price elasticities can become only smaller at lower prices. Hence, when a monopolistic variety becomes competitively priced, an increase in its market share caused by a drop in the price could only be smaller compared to the case of CES. Hence, it has the same upper bound, $\theta < e$.

Remark: Without (A2), θ could be arbitrarily large (see Appendix B), which could generate stable cycles of any period, or robust chaotic attractors with any number of intervals. By imposing (A2), we abstain ourselves from showing more exotic dynamic behaviors.

Question #2: How does θ depend on L/F? For CES, it is independent of L/F.

Proposition 2: If $\zeta(\cdot)-1$ is monotone and log-concave over an interval containing $(\underline{z}^c, \underline{z}^m)$, θ is increasing in L/F. If one of the monotonicity and the log-concavity conditions is strict, θ is strictly increasing in L/F.
Corollary: Under the weak (strong) 2nd Law, the strict (weak) log-concavity of $\zeta(\cdot)-1$ is sufficient for θ being strictly increasing in L/F.

Remark: The log-concavity of $\zeta(\cdot)-1$ is weaker than the concavity of $\zeta(\cdot)-1$, and hence the concavity of $\zeta(\cdot)$. In fact, if $s(\cdot)$ is thrice-continuously differentiable, $\zeta(\cdot)-1$ is strictly log-concave iff

$$\zeta''(\cdot) < \frac{\left(\zeta'(\cdot)\right)^2}{\zeta(\cdot)-1},$$

that is, when $\zeta(\cdot)$ is "not too convex."

Intuition: Under (A1), a higher L/F leads to an increase in both $\underline{z}^m$ and $\underline{z}^c$. Under (A2), this leads to an increase in both $\zeta(\underline{z}^m)$ and $\zeta(\underline{z}^c)$. The former implies a lower markup rate, and hence the price drop due to the loss of monopoly is smaller, which contributes to a smaller θ. The latter implies the market share responds more to the price drop, which contributes to a larger θ. If $\zeta(\cdot)$ is "not too convex," the former effect could not dominate the latter, so that θ is increasing in L/F.

Two Parametric Families: Generalized Translog and Constant Pass-Through

Ex.1: Generalized Translog: For $\beta > 0, \gamma > 0;\ \sigma > 1;\ \eta > 0;$

$$s(z) = \gamma\left(1 - \frac{\sigma-1}{\eta}\log\left(\frac{z}{\beta}\right)\right)^{\eta} = \gamma\left(\frac{1-\sigma}{\eta}\log\left(\frac{z}{\bar{z}}\right)\right)^{\eta} \quad \text{for } z < \bar{z} \equiv \beta e^{\frac{\eta}{\sigma-1}},$$

$$\zeta(z) = 1 + \frac{\sigma-1}{1 - \frac{\sigma-1}{\eta}\log\left(\frac{z}{\beta}\right)} = 1 - \frac{\eta}{\log\left(\frac{z}{\bar{z}}\right)} > 1, \quad \text{for } z < \bar{z} \equiv \beta e^{\frac{\eta}{\sigma-1}}$$

is strictly increasing in $z \in (0, \bar{z})$ with the range $(1, \infty)$, and hence satisfying (A2).

- Translog is a special case, where $\eta = 1$; CES is the limit case, where $\eta \to \infty$, while holding $\beta > 0$ and $\sigma > 1$ fixed,
- $\zeta(\underline{z}^m)$ is strictly increasing in $L/F \in (0, \infty)$ with the range, $(1, \infty)$.

$$\theta \equiv \frac{s(\underline{z}^c)}{s(\underline{z}^m)} = \left[1 + \frac{1}{\eta}\log\left[1 - \frac{1}{\zeta(\underline{z}^m)}\right]^{1-\zeta(\underline{z}^m)}\right]^{\eta} < \left(1 + \frac{1}{\eta}\right)^{\eta} < e,$$

strictly increasing in $L/F \in (0, \infty)$ with the upper bound $\left(1 + \frac{1}{\eta}\right)^{\eta} \to e$, as $\eta \to \infty$.

Notes:

- $\zeta(z) - 1$ not log-concave. This example shows that the log-concavity is *not necessary* for θ to be increasing in L/F.
- With $\eta > 1$, $\theta > 2$ for a sufficiently large L/F and hence $\delta(\theta - 1) > 1$ for a sufficiently large δ, causing the instability of the steady state. As $\eta \to \infty, \theta \to e$ for a sufficiently large L/F. Yet, in the CES limit, $\theta = (1 - 1/\sigma)^{1-\sigma} < e$. Thus, the dynamical system is discontinuous, as $\eta \to \infty$.
- Generalized Translog has counterfactual implications; the pass-through rate, $\partial \ln p^m / \partial \ln \psi$, is decreasing in the price.

Ex.2: Constant Pass-Through: For $\beta > 0; \gamma > 0;\ \sigma > 1$, let $0 < \rho = 1/(1+\Delta) < 1$ and $\Delta = (1-\rho)/\rho > 0$,

$$s(z) = \gamma\left[\sigma - (\sigma-1)\left(\frac{z}{\beta}\right)^{\Delta}\right]^{1/\Delta} = \gamma\sigma^{1/\Delta}\left[1 - \left(\frac{z}{\bar{z}}\right)^{\Delta}\right]^{1/\Delta} \qquad \text{with} \qquad \bar{z} \equiv \beta\left(\frac{\sigma}{\sigma-1}\right)^{1/\Delta} = \beta\left(\frac{\sigma}{\sigma-1}\right)^{\rho/(1-\rho)}$$

$$\Rightarrow \zeta(z) = \frac{1}{1-\left(1-\frac{1}{\sigma}\right)\left(\frac{z}{\beta}\right)^{\Delta}} = \frac{1}{1-\left(\frac{z}{\bar{z}}\right)^{\Delta}} > 1$$

strictly increasing in $z \in (0, \bar{z})$ with the range $(1, \infty)$, and hence satisfying (A2).

- $p^m = (\bar{z})^{1-\rho}(\psi)^{\rho} = (\beta)^{1-\rho}(\sigma\psi/(\sigma-1))^{\rho}$ → the pass-through rate, $\partial \ln p^m / \partial \ln \psi = \rho < 1$, is constant.
- CES is the limit case, as $\rho \to 1$ or $\Delta \to 0$, while holding $\beta > 0$ and $\sigma > 1$ fixed.
- $\underline{z}^m$ and $\underline{z}^c$ both strictly increasing in L/F, with $\underline{z}^m \to \bar{z}$ and $\underline{z}^c \to \bar{z}$, as $L/F \to \infty$.

$$\theta \equiv \frac{s(\underline{z}^c)}{s(\underline{z}^m)} = \left[\frac{1-\left(\underline{z}^c/\bar{z}\right)^{\Delta}}{1-\left(\underline{z}^m/\bar{z}\right)^{\Delta}}\right]^{1/\Delta} = \left[\frac{1-\left(\underline{z}^m/\bar{z}\right)^{\Delta(1+\Delta)}}{1-\left(\underline{z}^m/\bar{z}\right)^{\Delta}}\right]^{1/\Delta} < [1+\Delta]^{1/\Delta} < e$$

strictly increasing in L/F, with the upper bound, $[1+\Delta]^{1/\Delta} \to e$, as $\Delta \to 0$.

Notes:

- $\zeta(z) - 1$ not log-concave. This example shows that the log-concavity is *not necessary* for θ to be increasing in L/F.
- With $0 < \Delta < 1 \leftrightarrow 1/2 < \rho < 1$, $\theta > 2$ for a sufficiently large L/F and hence $\delta(\theta - 1) > 1$ for a sufficiently large δ, causing the instability of the steady state.
- As $\rho \to 1$ or $\Delta \to 0$, $\theta \to e$ for a sufficiently large L/F. Yet, in the CES limit, $\theta = (1-1/\sigma)^{1-\sigma} < e$. Thus, the dynamical system is discontinuous, as $\rho \to 1$ or $\Delta \to 0$.

A Multi-Market Extension

A Multi-Market Extension: J markets, $j = 1,2, \dots, J$, with market size L_j.
In each market, the single consumption good produced by assembling the market-specific intermediate inputs in Ω_j, with H.S.A. CRS production function, characterized by $s_j(\cdot)$; with the innovation cost, F_j, and the survival rate, δ_j.

Possible Interpretations

- Identical Households, whose preferences are Cobb-Douglas, $\sum_{j=1}^{J} \beta_j \ln X_j$ with $\sum_{j=1}^{J} \beta_j = 1$. Then, $L_j = \beta_j L$.
- J types of consumers, with L_j: the total income of type-j consumers. "Types" can be their "tastes" or "locations", etc.

Dynamical System: Dynamics of innovations in different markets are decoupled, with each following

$$n_{j,t+1} = f_j(n_{j,t}) \equiv \begin{cases} f_{jL}(n_{j,t}) \equiv & \delta_j(\theta_j + (1-\theta_j)n_{j,t}) & for\ n_{j,t} < 1 \\ f_{jH}(n_{j,t}) \equiv & \delta_j n_{j,t} & for\ n_{j,t} > 1 \end{cases}$$

where θ_j is given by

$$\theta_j \equiv \frac{s_j\left(z\left[1 - \frac{1}{\zeta_j(z)}\right]\right)}{s_j(z)} > 1; \ \frac{\zeta_j(z)}{s_j(z)} \equiv \frac{\beta_j}{F_j} L; \ \zeta_j(z) \equiv 1 - \frac{z s_j'(z)}{s_j(z)}.$$

Suppose that markets differ only in market size. Because innovation/entry activities fluctuate more in larger markets,

- They are not always higher in larger markets.
- The sale of each product is more volatile in larger markets.

Concluding Remarks

- In existing models of ***Endogenous Innovation Cycles***, m
  - Market size alters the amplitude of fluctuations without affecting the patterns of fluctuations.
  - Under **CES demand system**, an **exogenous markup rate** → ***Procompetitive effect*** of market size missing

- We extended the Judd model of endogenous innovation cycles, using **H.S.A.** to allow for the procompetitive effect.
  - Under H.S.A., the price elasticity of demand for each product, $\zeta(\cdot)$, depends solely on its "own relative price." If increasing, the procompetitive effect.
  - Dynamics, still generated by **the skewed-V shape map**, but its parameter, *the delayed impact of innovation*, depends on the market size/innovation cost ratio, L/F.

- **A larger L/F has destabilizing effects through the procompetitive effect.**
  - Under the log-concavity of $\zeta(\cdot) - 1$, i.e., $\zeta(\cdot)$ is "not too convex."
  - In two parametric families with choke prices, "Generalized Translog" and "Constant Pass-Through." Each contains CES as a limit case, and yet qualitative properties are discontinuous in the CES limit.

- In a multi-market extension, because innovation/entry activities fluctuate more in larger markets,
  - They are not always higher in larger markets.
  - The sale of each product is more volatile in larger markets.

- Monopolistic competition under H.S.A. should be valuable extensions to CES
  - Tractable and flexible
  - Two parametric families we introduced should also be useful for many applications.